# Equation of a Circle, Centre (0, 0) and Radius $r$

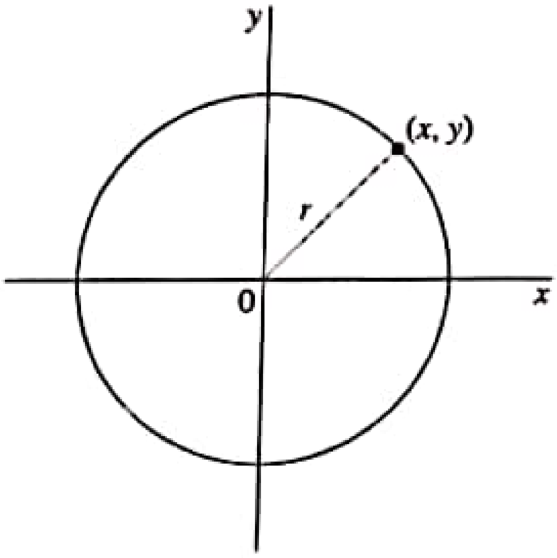
A circle is a set of points (a locus) which are equidistant from a fixed point called the '**centre**'.
The distance from the centre to any point on the circle is called the '**radius**'.
On the right is a circle with centre (0, 0), radius $r$ and $(x, y)$ any point on the circle.
Distance between (0, 0) and $(x, y)$ equals the radius, $r$.

$\therefore \quad \sqrt{(x-0)^2+(y-0)^2}=r$ (distance formula)

$\sqrt{x^2+y^2}=r$

$x^2+y^2=r^2$ (square both sides)

Hence, $x^2+y^2=r^2$ is said to be the equation of the circle.




| Equation of a circle, centre (0, 0) and radius $r$, is $x^2+y^2=r^2$. |
|---|

**Two quantities are needed to find the equation of a circle:**

| **1.** Centre | **2.** Radius |
|---|---|
| If the centre is (0, 0), the equation of the circle will be of the form $x^2+y^2=r^2$. | |

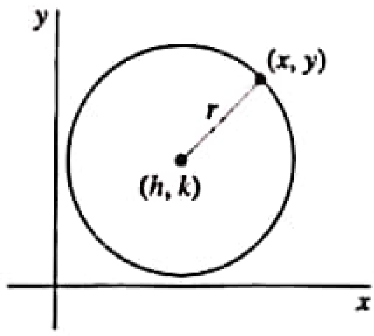
# Equation of a Circle, Centre $(h, k)$ and Radius $r$

On the right is a circle with centre $(h, k)$ and radius $r$, and $(x, y)$ is any point on the circle.
Distance between $(h, k)$ and $(x, y)$ equals the radius, $r$.

$\therefore \sqrt{(x-h)^2+(y-k)^2}=r$ (distance formula)

$(x-h)^2+(y-k)^2=r^2$ (square both sides)

Hence, $(x-h)^2+(y-k)^2=r^2$ is said to be the equation of the circle.




> The equation of a circle, centre $(h, k)$ and radius $r$, is
> $(x-h)^2+(y-k)^2=r^2$.

**Two quantities are needed to find the equation of a circle:**

> **1.** Centre, $(h, k)$ **2.** Radius, $r$
> Then use the formula $(x-h)^2+(y-k)^2=r^2$.

Note: If $(h, k)=(0, 0)$, the equation $(x-h)^2+(y-k)^2=r^2$ reduces to $x^2+y^2=r^2$.

### *Example*

**(i)** Find the centre and radius of the circle $(x-2)^2+(y+5)^2=9$.

**(ii)** Find the equation of the circle, centre $(1, -4)$ and radius $\sqrt{13}$.

**Solution:**

**(i)** $(x-2)^2+(y+5)^2=9$

Compare exactly to:

$(x-h)^2+(y-k)^2=r^2$

↓ ↓ ↓

$(x-2)^2+(y+5)^2=9$

$\therefore h=2, \quad k=-5, \quad r=3$

Thus, centre $=(2, -5)$ and radius $=3$.

**(ii)** Centre $=(1, -4)$, radius $=\sqrt{13}$

$h=1, \quad k=-4, \quad r=\sqrt{13}$

$(x-h)^2+(y-k)^2=r^2$

$(x-1)^2+(y+4)^2=(\sqrt{13})^2$

$(x-1)^2+(y+4)^2=13$

# General Equation of a Circle

The general equation of a circle is written as:

$$x^2+y^2+2gx+2fy+c=0$$

When the equation of a circle is given in this form, we use the following method to find its centre and radius.

1. Make sure every term is on the left-hand side and the coefficients of $x^2$ and $y^2$ are equal to 1.
2. Centre $=(-g,-f)=(-\frac{1}{2}$ coefficient of $x, -\frac{1}{2}$ coefficient of $y)$
3. Radius $=\sqrt{g^2+f^2-c}$ (provided $g^2+f^2-c>0$)

Notes: 1. The equation is of the second degree (highest power is 2).
2. The coefficients of $x^2$ and $y^2$ are equal.
3. There is no $xy$ term.

*Example*

Find the centre and radius of each of the circles:

(i) $x^2+y^2-4x+2y-11=0$ (ii) $x^2+y^2-8y+3=0$.

**Solution:**

(i) $x^2+y^2-4x+2y-11=0$

Centre $=(2,-1)$

Radius $=\sqrt{(2)^2+(-1)^2+11}$

$=\sqrt{4+1+11}=\sqrt{16}=4$

(ii) $x^2+y^2+0x-8y+3=0$ (put in $0x$)

Centre $=(0,4)$

Radius $=\sqrt{(0)^2+(4)^2-3}$

$=\sqrt{0+16-3}=\sqrt{13}$

**Example**

The equation of a circle with radius 5 is $x^2+y^2-6x+4ky+20=0$, $k \in \mathbf{Z}$.

**(i)** Find the centre of the circle and the radius length in terms of $k$.

**(ii)** Find the values of $k$.

**Solution:**

**(i)** $x^2+y^2-6x+4ky+20=0$

Centre $=(3,-2k)$

Radius $=\sqrt{(3)^2+(-2k)^2-20}$

$=\sqrt{9+4k^2-20}$

$=\sqrt{4k^2-11}$

**(ii)** Given: Radius $=5$

$\therefore \sqrt{4k^2-11}=5$

$4k^2-11=25$

$4k^2=36$

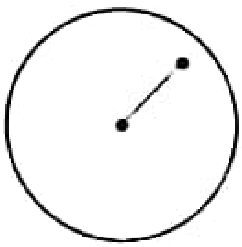
$k^2=9$

$k=\pm\sqrt{9}=\pm 3$

Points inside, on or outside a circle

**Method 1:**

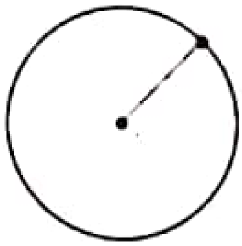
To find whether a point is inside, on or outside a circle, calculate the distance from the centre to the point and compare this distance with the radius. Three cases arise:



**Inside**

**1.** Distance from the centre, to the point is **less** than the radius

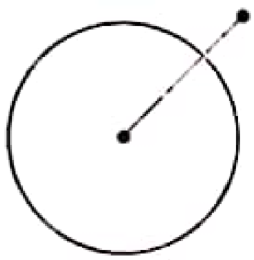
$\therefore$ point inside the circle.



**On**

**2.** Distance from the centre to the point is **equal** to the radius

$\therefore$ point on the circle.



**Outside**

**3.** Distance from the centre to the point is **greater** than the radius

$\therefore$ point outside the circle.

**Method 2:**

The equation of a circle can be of the form:

$$x^2+y^2=r^2$$

$$(x-h)^2+(y-k)^2=r^2$$

$$x^2+y^2+2gx+2fy+c=0$$

If the coordinates of a point satisfy the equation of a circle, then the point is **on** the circle. Otherwise, the point is either **inside** or **outside** the circle. By substituting the coordinates into the equation of the circle, one of the following situations can arise:

1. LHS < RHS: the point is **inside** the circle.
2. LHS = RHS: the point is **on** the circle.
3. LHS > RHS: the point is **outside** the circle.

# Parametric Equations of a Circle

A circle may be defined by a pair of parametric equations. On our course we shall meet two types of parametric equation.

## 1. Trigonometric Parametric Equations

$$x = h \pm r\cos\theta, \qquad y = k \pm r\sin\theta$$

are the parametric equations of the circle

$$(x-h)^2 + (y-k)^2 = r^2.$$

We use the fact that $\cos^2\theta + \sin^2\theta = 1$.

*Example*

The parametric equations of a circle are $x = -2 + \sqrt{3}\cos\theta$, $y = 1 + \sqrt{3}\sin\theta$.
Find its Cartesian equation. Find its centre and radius length.

**Solution:**

$$x = -2 + \sqrt{3}\cos\theta$$
$$x + 2 = \sqrt{3}\cos\theta$$
$$\frac{x+2}{\sqrt{3}} = \cos\theta$$

$$y = 1 + \sqrt{3}\sin\theta$$
$$y - 1 = \sqrt{3}\sin\theta$$
$$\frac{y-1}{\sqrt{3}} = \sin\theta$$

Thus, $\left(\frac{x+2}{\sqrt{3}}\right)^2 + \left(\frac{y-1}{\sqrt{3}}\right)^2 = \cos^2\theta + \sin^2\theta$

$$\frac{(x+2)^2}{3} + \frac{(y-1)^2}{3} = 1 \qquad (\cos^2\theta + \sin^2\theta = 1)$$

$$(x+2)^2 + (y-1)^2 = 3 \qquad \text{(multiply both sides by 3)}$$

This is the Cartesian equation of the circle, as it is in the form $(x-h)^2 + (y-k)^2 = r^2$.
The centre is $(-2, 1)$ and the radius is $\sqrt{3}$.